The Degree/Diameter Problem for Mixed Abelian Cayley Graphs

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Abstract

This paper investigates the upper bounds for the number of vertices in mixed abelian Cayley graphs with given degree and diameter. Additionally, in the case when the undirected degree is equal to one, we give a construction that provides a lower bound.

Keywords: Network design, Degree/Diameter problem, abelian Cayley graphs
2010 MSC: 05C30, 05C35

1. Introduction

The Degree/Diameter Problem asks for constructing the largest possible graph (in terms of the number of vertices), for a given maximum degree and a given diameter. The Degree/Diameter Problem can be formulated for directed, undirected graphs, or mixed graphs. In the case of directed graphs, if we denote by $\overrightarrow{N}_{d,k}$ the order of the largest possible digraph that can be

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constructed, we get the following upper bound:

$$\overrightarrow{N}_{d,k} \leq \overrightarrow{M}_{d,k} = 1 + d + d^2 + \cdots + d^k = \begin{cases} \frac{d^{k+1}-1}{d-1} & \text{if } d > 1 \\ k+1 & \text{if } d = 1 \end{cases} \tag{1}$$

where d is the maximum out-degree, and k is the diameter. The number $\overrightarrow{M}_{d,k}$ is called the *(directed) Moore bound*. In the case of undirected graphs, the Moore bound becomes

$$M_{\Delta,k} = 1+\Delta+\Delta(\Delta-1)+\cdots+\Delta(\Delta-1)^{k-1} = \begin{cases} 1+\Delta\ \frac{(\Delta-1)^k-1}{\Delta-2} & \text{if } \Delta > 2 \\ 2k+1 & \text{if } \Delta = 2 \end{cases} \tag{2}$$

where Δ is the maximum degree, and k represents the diameter. These latter bounds are easily derived just counting the number of vertices of a particular distance of any given vertex v in a [di]graph with given maximum [out-]degree and diameter [13].

In mixed graphs we have both arcs and edges. Thus we have three parameters: a maximum undirected degree r, a maximum directed out-degree z, and diameter k. The upper bound was first stated as

$$1 + (r+z) + \cdots + \left[z(r+z)^{k-1} + r(r+z-1)^{k-1}\right] \tag{3}$$

but it was recently adjusted ([3]) to

$$M_{z,r,k} = A\frac{u_1^{k+1}-1}{u_1-1} + B\frac{u_2^{k+1}-1}{u_2-1} \tag{4}$$

where

$$\begin{aligned} v &= (z+r)^2 + 2(z-r) + 1 \\ u_1 &= \frac{z+r-1-\sqrt{v}}{2} \\ u_2 &= \frac{z+r-1+\sqrt{v}}{2} \\ A &= \frac{\sqrt{v}-(z+r+1)}{2\sqrt{v}} \\ B &= \frac{\sqrt{v}+(z+r+1)}{2\sqrt{v}} \end{aligned}$$

Besides these general bounds given above, researchers are also interested in some particular versions of the problem, namely when the graphs are restricted to a certain class, such as the class of bipartite graphs [8], planar graphs [6, 18], vertex-transitive graphs [11, 20], Cayley graphs [11, 20, 19], Cayley graphs of abelian groups [5], or circulant graphs [21, 14, 7, 9]. In this paper we are concerned with mixed abelian Cayley graphs.

For most of these graph classes there exist Moore-like upper bounds, which in general are smaller than the Moore bound for general graphs, although some of them are quite close to the Moore bound. For example, the best known general upper bound for undirected vertex-transitive and Cayley graphs is $M_{\Delta,k}-2$ (see [13]). On the other hand, the upper bounds for abelian Cayley graphs and digraphs are significantly smaller. Let $\overrightarrow{N}_{d,k}^{AC}$ be the number of vertices of the largest abelian Cayley digraph with degree d and diameter k; then

$$\overrightarrow{N}_{d,k}^{AC} \leq \binom{k+d}{d} = \binom{k+d}{k} \tag{5}$$

(see [5]).

The generating function of the sequence $\binom{k+d}{k}$ is

$$\overrightarrow{A}_d(s) = \frac{1}{(1-s)^{d+1}}. \tag{6}$$

Alternatively, the upper bound for undirected abelian Cayley graphs is a bit more involved. It was proved in [2] that, if $\Delta = 2t$, then

$$N_{\Delta,k}^{AC} \leq F_{t,k} = \sum_{i=0}^{t} 2^i \binom{t}{i}\binom{k}{i} \tag{7}$$

The numbers $F_{t,k}$ of Equation 7 are known as *Delannoy numbers* (sequence A008288 of [1]), and they arise in a variety of combinatorial and geometric problems [17]. For example, they correspond to the volume of the ball of radius $k/2$ in the L^1 metric in t dimensions [5, 12]. They satisfy the recurrence

$$\begin{aligned} &F_{t,k} = F_{t-1,k} + F_{t,k-1} + F_{t-1,k-1}, \text{ with} \\ &F_{t,1} = 2t+1, \quad \text{for } t \geq 0. \end{aligned} \tag{8}$$

Other exact and asymptotic formulas are given in [16, 12, 9], such as

$$F_{t,k} = \sum_{i=0}^{t} \binom{t}{i} \binom{k+i}{t} = \sum_{i=0}^{t} \binom{k+i}{i} \binom{k}{t-i} \tag{9}$$

Delannoy numbers are symmetric, hence we can swap t and k in Equation 9, to get

$$F_{t,k} = \sum_{i=0}^{k} \binom{k}{i} \binom{t+i}{k} = \sum_{i=0}^{k} \binom{t+i}{i} \binom{t}{k-i} \tag{10}$$

The generating function for the numbers $F_{t,k}$, as specified in Equation 10, is given in [12]. It is

$$A_t(s) = \frac{(1+s)^t}{(1-s)^{t+1}}. \tag{11}$$

A detailed discussion of the case $\Delta = 2t + 1$ can be found in [9].

Even though the abelian property of the underlying group prevents abelian Cayley graphs (either directed or undirected) to grow as large as their non-abelian counterparts, these graphs have been widely used as topologies for computer networks and parallel computers, due to their other nice properties. Paraphrasing [5]: "…the extra structure provided by the groups may provide compensating advantages …, such as good routing algorithms, easy constructibility, and the ability to map common problems onto the architecture".

As for mixed graphs, they also arise in many practical situations, urban street networks being perhaps the most obvious one. Some recent studies discuss mixed graphs as models for interconnection networks [4]. It is therefore natural to consider network topologies based on mixed abelian Cayley graphs, and investigate the Degree/Diameter Problem in that kind of graphs.

The remainder of the paper is divided into three sections: Section 2 deals with the general upper bound, and Section 3 describes the aforementioned construction. Finally, Section 4 states a few open problems.

2. Upper bound for mixed abelian Cayley graphs

Let Γ be an abelian group, and let Σ be a generating set of Γ containing r_1 involutions and r_2 pairs of generators and their inverses, and z additional generators, whose inverses are not in Σ. Thus, the Cayley graph $Cay(\Gamma, \Sigma)$ is a mixed graph with undirected degree r, where $r = r_1 + 2r_2$, and directed out-degree z. The following theorem gives an upper bound for the number of vertices of $Cay(\Gamma, \Sigma)$, as a function of its diameter:

Theorem 1. *Let Γ and Σ be as before. The number of vertices of the Cayley graph $Cay(\Gamma, \Sigma)$ is bounded above by*

$$\sum_{i=0}^{k} \binom{r_2 + z + i}{i} \binom{r_1 + r_2}{k - i}, \tag{12}$$

where k represents the diameter of $Cay(\Gamma, \Sigma)$.

Proof: The order of $Cay(\Gamma, \Sigma)$ is bounded above by the number of reduced words of length at most k, that can be formed in an arbitrary abelian monoid Γ, with a generating set Σ consisting of r_1 involutions and r_2 pairs of generators and their inverses, and z additional generators, whose inverses are not in Σ. More formally,

$$\Sigma = \{\iota_1, \ldots, \iota_{r_1}, \kappa_1, \kappa_1^{-1}, \ldots, \kappa_{r_2}, \kappa_{r_2}^{-1}, \delta_1, \ldots, \delta_z\}$$

In this setting, a word is reduced if it does not contain any subword of the form ι_i^2, or $\kappa_j \kappa_j^{-1}$. We denote by $P_{r_1,r_2,z,k}$ the number of such reduced words.

Now let $r_1 > 0$, then we can decompose the set of reduced words into two subsets: words containing ι_1, and words not containing ι_1. These two subsets have essentially the same structure as their parent set, and hence their respective cardinalities are $P_{r_1-1,r_2,z,k}$ and $P_{r_1-1,r_2,z,k-1}$. Thus, we get the following recurrence equation for $P_{r_1,r_2,z,k}$:

$$P_{r_1,r_2,z,k} = P_{r_1-1,r_2,z,k} + P_{r_1-1,r_2,z,k-1}, \tag{13}$$

We introduce $A_{r_1,r_2,z}(s) = \sum_{k=0}^{\infty} P_{r_1,r_2,z,k}\, s^k$, the generating function associated with the number sequence $P_{r_1,r_2,z,k}$, where $s \in \mathbb{C}$. By Equation 13 we have that,

$$A_{r_1,r_2,z}(s) = (1 + s)^{r_1} A_{0,r_2,z}, \tag{14}$$

hence this function depends on $A_{0,r_2,z}$.

Now let $r_1 = 0$. From here on we denote $P_{0,r_2,z,k}$ as $P_{r_2,z,k}$ in order to simplify the notation. In this case we can decompose the set of reduced words into three subsets: words containing κ_1, words containing κ_1^{-1}, and words not containing κ_1 or κ_1^{-1}. If a word contains κ_1, then it can contain further occurrences of κ_1, but it cannot contain κ_1^{-1}. In the same manner, if a word contains κ_1^{-1}, then it can contain further occurrences of κ_1^{-1}, but it cannot contain κ_1. Hence this time the cardinality of the first two subsets cannot simply be computed as $P_{r_2-1,z,k}$, because these two subsets include an incomplete pair of generator and inverse.

Accordingly, we denote the cardinality of the first two subsets as $S_{r_2,z,k-1}$, where $S_{r_2,z,k-1}$ denotes the set of reduced words of length $k-1$ taken from an alphabet $\Sigma' \subset \Sigma$ consisting of 0 involutions, generator κ_1, or alternatively κ_1^{-1}, $r_2 - 1$ pairs of generators and their respective inverses, and all z additional generators whose inverses are not in Σ. With this notation we get the following system of recurrence equations for the number of vertices:

$$\begin{aligned} P_{r_2,z,k} &= 2S_{r_2,z,k-1} + P_{r_2-1,z,k} \\ S_{r_2,z,k-1} &= S_{r_2,z,k-2} + P_{r_2-1,z,k-1} \end{aligned} \tag{15}$$

with $P_{r_2,z,0} = 1$, $S_{r_2,z,1} = 2r_2 + z$, and $P_{0,z,k} = \binom{k+z}{z}$. Note that $S_{r_2,z,k}$ only plays the role of an auxiliary element.

Substituting for $S_{r_2,z,k-1}$ in the first equation of 15, we get

$$\begin{aligned} P_{r_2,z,k} &= 2S_{r_2,z,k-2} + 2P_{r_2-1,z,k-1} + P_{r_2-1,z,k} \\ S_{r_2,z,k-2} &= S_{r_2,z,k-3} + P_{r_2-1,z,k-2} \end{aligned}$$

After $i-1$ substitutions we get

$$\begin{aligned} P_{r_2,z,k} &= 2S_{r_2,z,k-i} + 2\sum_{j=1}^{i-1} P_{r_2-1,z,k-j} + P_{r_2-1,z,k}, \\ S_{r_2,z,k-i} &= S_{r_2,z,k-i-1} + P_{r_2-1,z,k-i}. \end{aligned}$$

This process ends when $i = k - 1$:

$$
\begin{aligned}
P_{r_2,z,k} &= 2S_{r_2,z,1} + 2\sum_{j=1}^{k-2} P_{r_2-1,z,k-j} + P_{r_2-1,z,k}, \\
&= 2S_{r_2,z,1} + 2\sum_{j=2}^{k-1} P_{r_2-1,z,j} + P_{r_2-1,z,k}, \\
S_{r_2,z,1} &= S_{r_2,z,k-i-1} + P_{r_2-1,z,-i}.
\end{aligned}
$$

Now,

$$P_{r_2,z,k-1} = 2S_{r_2,z,1} + 2\sum_{j=2}^{k-2} P_{r_2-1,z,j} + P_{r_2-1,z,k-1}.$$

Subtracting $P_{r_2,z,k-1}$ from $P_{r_2,z,k}$ we get

$$P_{r_2,z,k} = P_{r_2,z,k-1} + P_{r_2-1,z,k} + P_{r_2-1,z,k-1}, \tag{16}$$

with $P_{r_2,z,0} = 1$, and $P_{0,z,k} = \binom{k+z}{z}$.

Equation 16 is the well-known recurrence of Delannoy numbers, mentioned in Section 1. Now let $B_{r_2,z}(s) = A_{0,r_2,z}(s) = \sum_{k=0}^{\infty} P_{r_2,z}\, s^k$ be the generating function associated with the number sequence $P_{r_2,z,k}$, where $s \in \mathbb{C}$. It is easy to see that $B(s)$ satisfies

$$B_{r_2,z}(s) = \frac{1+s}{1-s} B_{r_2-1,z}(s)$$

With the aid of the boundary condition $P_{0,z,k} = \binom{k+z}{z} = \binom{k+z}{k}$ we get

$$B_{0,z}(s) = \sum_{k=0}^{\infty} \binom{k+z}{k} s^k = \frac{1}{(1-s)^{z+1}},$$

and hence

$$B_{r_2,z}(s) = \frac{(1+s)^{r_2}}{(1-s)^{r_2+z+1}} \tag{17}$$

We can now give the general expression for the generating function of Equation 14:

$$A_{r_1,r_2,z}(s) = (1+s)^{r_1} B_{r_2,z} = \frac{(1+s)^{r_1+r_2}}{(1-s)^{r_2+z+1}} \tag{18}$$

Since $A_{r_1,r_2,z}(s)$ is the product of the functions $(1+s)^{r_1+r_2} = \sum_{k=0}^{\infty} \binom{r_1+r_2}{k} s^k$, and $\frac{1}{(1-s)^{r_2+z+1}} = \sum_{k=0}^{\infty} \binom{r_2+z+k}{k} s^k$, the general term for $A_{r_1,r_2,z}(s)$ can be obtained by convolution of the general terms of the corresponding factor series, and hence our result follows.

□

Remark 1.

$$\sum_{i=0}^{k} \binom{r_2+z+i}{i} \binom{r_1+r_2}{k-i} = \sum_{i=0}^{k} \binom{r_1+r_2}{i} \binom{r_2+z+k-i}{k-i}$$

□

Note also that Theorem 1 generalizes the upper bound for directed and undirected abelian Cayley graphs. Indeed, if we let $r_1 = r_2 = 0$ in Equation 18, and make $z = d$, we get Equation 6. In the same manner, letting $z = r_1 = 0$ and making $r_2 = t$, we get Equation 11.

It is well known that Delannoy numbers have no closed form, meaning that they cannot be represented as a linear combination of a fixed number of hypergeometric terms (something which can be verified with the aid of the methods developed in [15]). Therefore, the numbers $P_{r_1,r_2,z,k}$ that appear in the proof of Theorem 1 do not have a closed form either, since $P_{0,r_2,z,k}$ satisfies the same recurrence as Delannoy numbers. However, we can extract asymptotic information from the generating function $A_{r_1,r_2,z}(s)$ above. Recall that α is an *algebraic singularity* of the function f if f can be written near α as

$$f(s) = f_0(s) + \frac{g(s)}{(1-s/\alpha)^{\omega}} \tag{19}$$

where f_0 and g are analytic near α, g is nonzero near α, and ω is a real number different from $0, -1, -2, \ldots$. We readily recognize that 1 is an algebraic singularity of $A_{r_1,r_2,z}(s)$, since $A_{r_1,r_2,z}(s)$ can be written in the above form, with $g(s) = (1+s)^{r_1+r_2}$, and all the other conditions are satisfied. Now we can apply Theorem 3 of [10], that we reproduce here:

Theorem 2. *Suppose that for some real $\rho > 0$, $A(s)$ is analytic in the region $|s| < \rho$, and has a finite number $\tau > 0$ of singularities on the circle $|s| = \rho$,* *all of which are algebraic. Let α_i, ω_i, and g_i be the values of α, ω, and g in (19), corresponding to the i-th such singularity. Then $A(s)$ is the generating function for a sequence $\langle a_n \rangle$ satisfying*

$$a_n = \frac{1}{n} \sum_{i=1}^{\tau} \frac{g_i(\alpha_i) n^{\omega_i}}{\Gamma(\omega_i) \alpha_i^n} + o(\rho^{-n} n^{\Omega - 1})$$

where Ω is the maximum of the ω_i and Γ denotes the Gamma function.

Hence we obtain

Corollary 1.

$$P_{r_1,r_2,z,k} = \frac{2^{r_1+r_2} k^{r_2+z+1}}{k\Gamma(r_2+z+1)} + o(k^{r_2+z}) = \frac{2^{r_1+r_2} k^{r_2+z}}{(r_2+z)!} + o(k^{r_2+z}) \tag{20}$$

□

whence, with the aid of Stirling's approximation for the factorial $(r_2+z)!$, we get

Corollary 2.

$$P_{r_1,r_2,z,k} \sim \frac{2^{r_1+r_2}}{\sqrt{2\pi(r_2+z)}} \left(\frac{ek}{r_2+z} \right)^{r_2+z} \tag{21}$$

□

Thus, for mixed abelian Cayley graphs we get an upper bound that is significantly lower, asymptotically, than the best known general upper bound for mixed graphs. Indeed, if we fix the directed and undirected degrees, we can see that Equation 3 grows exponentially as a function of k, whereas Equation 12 only grows polynomially.

The generating function is not only useful for obtaining asymptotic estimates. We can also expand $A_{r_1,r_2,z}(s)$ to obtain all the mixed upper bounds for abelian Cayley graphs up to a given diameter:

The first interesting case is $k = 2$. Taking that $r = r_1 + 2r_2$ and $d = r + z$, we can simplify this upper bound as:

$$1 + r_2 + z + \frac{1}{2}\left(d^2 + d\right) \tag{22}$$

We can compare this bound with the corresponding upper bound for the general case, which is $1 + z + d^2$. Asymptotically, as the total degree d approaches infinity, the ratio between Equation 22 and $1 + z + d^2$ approaches $1/2$.

diameter k	Upper bound for abelian Cayley graphs
0	1
1	$1+r_1+2r_2+z$
2	$\frac{1}{2}\left(2+r_1^2+4r_2^2+3z+z^2+4r_2(1+z)+r_1(1+4r_2+2z)\right)$
$\vdots$	$\vdots$

3. Approaching the mixed abelian Cayley upper bound for undirected degree $r = 1$

In this section we deal with the problem of finding families of abelian Cayley graphs with given degree and diameter, and order approaching the upper bound. Circulant graphs are Cayley graphs over $\mathbb{Z}_n$, and they have been studied for this problem for both the directed and the undirected case. As in the general case, the definition of circulant graph can be extended to allow both edges and arcs. Let Σ be a generating set of $\mathbb{Z}_n$ containing r_1 involutions and r_2 pairs of generators, together with their inverses, and z additional generators, whose inverses are not in Σ. The *(mixed) circulant graph* $C(n;\Sigma)$ has vertex set $V = \mathbb{Z}_n$, and each vertex i is connected to $i+a \pmod n$, for all $a \in \Sigma$. Thus, $C(n;\Sigma)$ has undirected degree r, where $r = r_1 + 2r_2$, and directed degree z. In fact, $r_1 \leq 1$, since $\mathbb{Z}_n$ has either one involution (n even), or none (n odd).

We will now approach the upper bound for abelian Cayley graphs in the simple case $r = 1$ with the aid of circulant graphs. If $r = 1$ and $z = 1$, Equation 12 becomes

$$\sum_{i=0}^{k} \binom{1}{i}\binom{1+k-i}{k-i} = 1+2k$$

This bound is unattainable, since the subgraph induced by the (undirected) edges of any mixed graph with $r = 1$ is a disjoint union of complete graphs of order 2, and hence has even order. So the upper bound becomes $2k$. Now, the (mixed) circulant graph $C(2k;\Sigma)$ has trivially diameter k (see Figure 1) when $\Sigma = \{1, k\}$.

More generally, for $r = 1$ and $z \geq 1$ the upper bound becomes

$$\sum_{i=0}^{k} \binom{1}{i}\binom{z+k-i}{k-i} = \frac{2k+z}{k+z}\binom{k+z}{k}$$

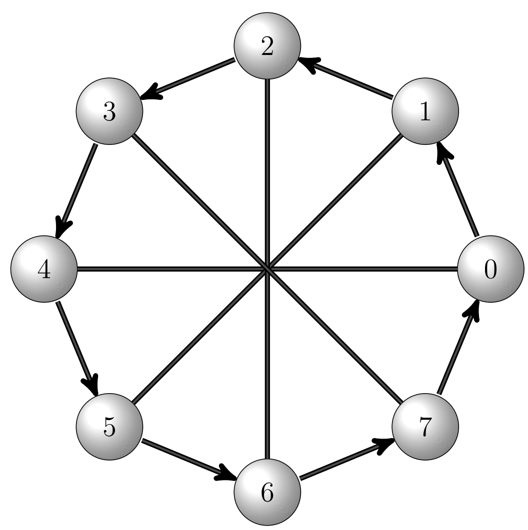

Figure 1: The (mixed) circulant graph $C(8;\{1,4\})$ has undirected degree $r=1$, directed degree $z=1$ and diameter $k=4$. This mixed graph achieves the largest possible order for a mixed abelian Cayley graph with these parameters (r,z,k).

which is asymptotically close to $\dfrac{2k^z}{z!}$ for large k (see Corollary 1). Next, we give an approximation of the latter bound by constructing a mixed circulant graph with order $(1+\frac{2k-1}{2z-1})^z$. We extend the construction of Wong and Coppersmith (see [21]) by adding an involution.

Proposition 1. *For every $z \geq 1$ and every even $n > 2$, the diameter of the mixed circulant graph $C(n^z; \{1, n, n^2, \ldots, n^{z-1}, \frac{1}{2}n^z\})$ is $(z-1)(n-1)+\frac{n}{2}$.*

Proof: Let V be the set of vertices of $C(n^z; \{1, n, n^2, \ldots, n^{z-1}, \frac{1}{2}n^z\})$, which contains all the integers l such that $0 \leq l < n^z$. Let us consider $(l_0, l_1, \ldots, l_{z-1})$ as the vector corresponding to the components of l in base n, that is,

$$l = l_0 + l_1 n^1 + l_2 n^2 + \cdots + l_{z-1} n^{z-1},$$

where $0 \leq l_i < n$ for each $i \in \{0, 1, \ldots, z-1\}$. There is a natural isomorphism, given by the mapping $l \mapsto (l_0, l_1, \ldots, l_{z-1})$, between V and the set of vertices of the cartesian product of $z-1$ circulant graphs $C(n; \{1\})$ and $C(n; \{1, \frac{n}{2}\})$, that is, the mixed graph

$$C(n; \{1\}) \square \cdots \square C(n; \{1\}) \square C(n; \{1, \frac{n}{2}\}).$$

As a consequence, the diameter of $C(n^z; \{1, n, n^2, \ldots, n^{z-1}, \frac{1}{2}n^z\})$ is precisely the sum of the diameters of the factors of such cartesian product, that is, $(z-1)(n-1)+\frac{n}{2}$. □

We have also computed the smallest diameter $k_{\min}$ for a circulant graph of order n^2 with two distinct generators, plus the involution $\frac{1}{2}n^2$. Let us denote by $\{a, b\}_{\min}$ a pair of generators that gives minimal diameter. We can see in Table 1 that the diameter of $C(n^2; \{1, n, \frac{1}{2}n^2\})$ stays close to the minimum for relatively large values of the order.

More generally, taking $k = z(n-1) - (\frac{1}{2}n - 1)$, and writting n in terms of k and z we have

Corollary 3. *Let k and z be positive integers such that $\frac{2k-1}{2z-1}$ is an odd integer at least 3. Then, there is a (mixed) circulant graph with parameters $r = 1$, $z \geq 1$ and diameter k, with order $n = (1+\frac{2k-1}{2z-1})^z$.*

Order n^2	$\frac{3}{2}n-1$	$k_{\min}$	$\{a,b\}_{\min}$
16	5	4	$\{1,3\}$
36	8	7	$\{1,4\}$
64	11	10	$\{1,5\}$
100	14	12	$\{1,14\}$
144	17	14	$\{1,20\}$
196	20	17	$\{1,27\}$
256	23	20	$\{1,14\}$

Table 1: A comparision between the diameter of $C(n^2;\{1,n,\frac{1}{2}n^2\})$ and the minimum diameter $k_{\min}$ for any circulant mixed graph of order n^2 with two distinct generators $\{a,b\}_{\min}$ plus the involution.

As a consequence, $C(n^z;\{1,n,n^2,\ldots,n^{z-1},\frac{1}{2}n^z\})$ approaches the upper bound asymptotically, as the diameter k increases:

$$\lim_{k\to\infty}\frac{(1+\frac{2k-1}{2z-1})^z}{\frac{2k+z}{k+z}\binom{k+z}{k}}=\frac{2^{z-1}\Gamma(z+1)}{(2z-1)^z}=\frac{2^{z-1}z!}{(2z-1)^z}$$

This means that for any value of the directed degree z, we have a construction that approaches the upper bound by a factor that is a function depending only on z. This aproximation is good only for small values of z. For instance, $C(n^2;\{1,n,\frac{1}{2}n^2\})$ approaches the upper bound by the factor $\frac{4}{9}$.

The diameter of circulant digraphs has been studied in depth. In particular, the generators $\{a,b\}$ of circulant digraphs $C(2n;\{a,b,n\})$ with minimum diameter are known for some values of n. Given the construction described in Proposition 1, it is natural to consider the following question:

If $\{a,b\}$ is a pair of generators that gives the minimum possible diameter for a circulant digraph of order $2n$, then is it true that the diameter of $C(2n;\{a,b,n\})$ is the minimum possible among all mixed circulant graphs of order $2n$, with two generators plus the involution n?

The answer is no in general, but the first counterexample appears at $n=19$ (i.e. a circulant graph of order 38). The minimum diameter for a circulant digraph of order 38 is 9, and there are 54 different generating sets $\{a,b\}$ that yield the minimum diameter. If we add the involution, the minimum diameter drops to 7, with 288 different generating sets yielding diameter 7. However, the intersection of the two families of optimal generating sets is empty.

4. Open problems

Although the mixed abelian Cayley graphs described in Section 3 have large order, it is quite plausible that other constructions can be found, yielding an even larger number of vertices. In particular, it could be interesting to have a family with order asymptotically approaching the upper bound.

Problem 1. *Construct (if possible) mixed abelian Cayley graphs of undirected degree $r = 1$ and order $n(k, z)$ such that*

$$\lim_{k\to\infty} \frac{n(k,z)}{\frac{2k+z}{k+z}\binom{k+z}{k}} = 1.$$

For the general problem, a number of constructions have been developed for either directed or undirected graphs, and some of them might be suitable for the mixed case (see, for example [5, 11, 20]). Nevertheless, other constructions, specific for the mixed case, might prove more effective.

Problem 2. *Find families of mixed abelian Cayley graphs with large order.*

In general, the upper bounds given in Section 2 are not tight. It is quite important to refine those bounds, both in general and for some specific cases.

Problem 3. *Provide sharper upper bounds for the number of vertices of a largest mixed abelian Cayley graph.*

It is quite interesting to see if there is a combination of values for which the order of the mixed abelian Cayley graphs is maximum, assuming that we fix the diameter k and the total degree $d = r_1 + 2r_2 + z$. Intuition and some numerical experiments suggest that Equation 21 reaches its maximum value at $r_1 = 0$ as k grows. It also seems that r_2 approaches 0 at the maximum, for large k. However, this has to be confirmed analytically, which leads to:

Problem 4. *Find the maximum of the function*

$$U_k(r_1, r_2, z) = \sum_{i=0}^{k} \binom{r_2 + z + i}{i}\binom{r_1 + r_2}{k - i},$$

for fixed k and fixed $d = r_1 + 2r_2 + z$.

In any case, even if the previous conjecture turned out to be true, it is still interesting to find the maximum of U_k for small values of k. As it may be difficult to work with the actual upper bound U_k, an alternative could be to work with its asymptotic approximation:

Problem 5. *Find the maximum of the function*

$$U'_k(r_1, r_2, z) = \frac{2^{r_1+r_2}}{\sqrt{2\pi(r_2+z)}} \left(\frac{ek}{r_2+z}\right)^{r_2+z}$$

for fixed k and fixed $d = r_1 + 2r_2 + z$.

The latter problem may be an interesting exercise in mathematical analysis. However, let us bear in mind that U'_k is only an asymptotic approximation of the actual upper bound, which in turn is not tight. Therefore, it would not be possible to draw any conclusions for the actual graphs.

Acknowledgements

The authors have been supported in part by grants MTM2010-21580-C02-01 and IPT-2102-0603-430000, from Ministerio Español de Economía y Competitividad, and by grant 2009SGR-442, from the Generalitat de Catalunya.

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